nodes, with utilization of the boundary conditions (Ref. 3).

However, the problem is closed if we take the arithmetic average of equations at the nodes. One obtains a valid equation at each node, as the resultant equation remains an integrated differential equation with a weighting function. Then the total number of equations and unknowns are both the number of nodes times the number of equations at each node. This completes the formulation.

In conclusion, it is noted that the advantage of a finite-element method lies in its flexibility in fitting a curved boundary. Whether the present formulations will be of practical value remains to be tested by actual programing of some examples.

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Accuracy of Donnell's Equations for Noncircular Cylinders

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Many papers have considered the stresses and displacements of noncircular cylindrical shells with a variety of edge conditions. Kempner et al.[1–5] have considered the cases of oval cylinders under lateral pressure for simply supported, clamped, and ring-reinforced end conditions where the radius of curvature relation for the oval cross section stated by Marguerre[6] was used as well as equations of the Donnell type. Basuli[7] presented a solution using Donnell type equations with the radius of curvature varying exponentially. It has been shown by Kempner,[8] Hoff,[9] and Morley,[10] however, that the results for circular cylinders when the Donnell equations are used are good only if the cylinder is short. Nevertheless, as Kraus has pointed out,[11] the accuracy of Donnell's assumptions for noncircular cylinders has not been assessed. It is the purpose of the present Note to perform this assessment.

Governing Equations

To investigate the applicability of the Donnell equations when considering noncircular cylindrical shells, the solution found using the equations of the Love-Reissner type will be compared with that obtained using the Donnell equations. The particular case considered for this comparison is an oval cylindrical shell that is simply supported at the edges and has a radius of curvature of the form stated by Marguerre[6]:

$$1/r = (1/r_o)(1 + \xi \cos 4\pi s/L_o) \tag{1}$$

where $r(s)$ is the radius of curvature of the cross section; r_o is the radius of the circle whose perimeter is L_o, the perimeter of the oval cross section; and ξ is the parameter that determines the noncircularity of the oval. To preclude the concave outward case, $0 \leqslant \xi \leqslant 1$. The coordinate system used for such shells is x, s, z where x is in the axial direction, s is in the circumferential direction, and z is the outward normal to the middle surface of the shell, as shown in Fig. 1.

The governing equations for thin cylindrical shells can be reduced to three involving u, v, and w—the translations of the middle surface of the shell in the x, s, and z directions, respectively. These equations are[11]

$$\frac{\partial^2 u}{\partial x^2} - \left(\frac{1-\nu}{2}\right)\frac{\partial^2 u}{\partial s^2} + \left(\frac{1+\nu}{2}\right)\frac{\partial^2 v}{\partial x \partial s} + \frac{\nu}{r}\frac{\partial w}{\partial x} = 0 \tag{2a}$$

$$(\partial^2 v/\partial s^2) + [(1-\nu)/2](\partial^2 v/\partial x^2) + [(1+\nu)/2](\partial^2 u/\partial x \partial s) + (\partial/\partial s)(w/r) + (1/r)(h^2/12)\{(\partial^2/\partial s^2)(v/r) + (1/r) \times [(1-\nu)/2](\partial^2 v/\partial x^2) - (\partial^3 w/\partial s^3) - (\partial^3 w/\partial x^2 \partial s)\} = 0 \tag{2b}$$

$$\nabla^4 w + (1/r)(12/h^2)[\nu(\partial u/\partial x) + (\partial v/\partial s) + (w/r)] - (\partial^3/\partial s^3)(v/r) - (\partial/\partial s)[(1/r)(\partial^2 v/\partial x^2)] = -12q(1-\nu^2)/Eh^3 \tag{2c}$$

when only a uniform lateral external pressure loading q is considered and where ν is the Poisson's ratio of the material, E is its Young's modulus, and h is the shell thickness. ∇^4 is the biharmonic operator.

These equations can be reduced to those attributed to Donnell by assuming that: 1) the transverse shear resultant makes a negligible contribution to the equilibrium of forces in the circumferential direction, and 2) the displacement v negligibly affects the changes in curvature and twist due to the loading. When these two assumptions are made, the three governing equations reduce to[11]

$$\frac{\partial^2 u}{\partial x^2} + \left(\frac{1-\nu}{2}\right)\frac{\partial^2 u}{\partial s^2} + \left(\frac{1+\nu}{2}\right)\frac{\partial^2 v}{\partial x \partial s} + \frac{\nu}{r}\frac{\partial w}{\partial x} = 0 \tag{3a}$$

$$\frac{\partial^2 v}{\partial s^2} + \left(\frac{1-\nu}{2}\right)\frac{\partial^2 v}{\partial x^2} + \left(\frac{1+\nu}{2}\right)\frac{\partial^2 u}{\partial x \partial s} + \frac{\partial}{\partial s}\left(\frac{w}{r}\right) = 0 \tag{3b}$$

$$\nabla^4 w - \frac{12}{rh^2}\left(\nu\frac{\partial u}{\partial x} + \frac{\partial v}{\partial s} + \frac{w}{r}\right) = -\frac{12q(1-\nu^2)}{Eh^3} \tag{3c}$$

Solution

To compare the two sets of equations—those of the Love-Reissner type with those of the Donnell type—a simply supported oval cylindrical shell is analyzed in detail. The boundary conditions at $x = 0$ and $x = L$ for such a situation are

$$w = v = \partial u/\partial x = \partial^2 w/\partial x^2 = 0 \tag{4}$$

With these boundary conditions, the solutions for the displacements can be assumed to be of the form

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \sum_{m=1,3,\ldots}^{\infty} \sum_{n=0,4,\ldots}^{\infty} \left(\frac{q r_o^2}{Eh}\right) \begin{Bmatrix} A_{mn} \cos\dfrac{m\pi x}{L} \cos\dfrac{n\pi s}{L_o} \\ B_{mn} \sin\dfrac{m\pi x}{L} \sin\dfrac{n\pi s}{L_o} \\ C_{mn} \sin\dfrac{m\pi x}{L} \cos\dfrac{n\pi s}{L_o} \end{Bmatrix} \tag{5}$$

when the loading is expanded in the form:

$$\frac{12q(1-\nu^2)}{Eh^3} = \sum_{m=1,3,\ldots}^{\infty} \left(\frac{q r_o^2}{Eh}\right)\left[\frac{48(1-\nu^2)}{m h^2 r_o^2}\right] \sin\frac{m\pi x}{L} \tag{6}$$

Because of the noncircularity of the cross section, it is not possible to assume an exponential form of solution and compare the roots of the complementary equation as was done for the circular case.[8–11] Therefore, the method of solution is to substitute the assumed solutions in Eqs. (2) and (3) and, using Eq. (1) for the radius of curvature relation, solve

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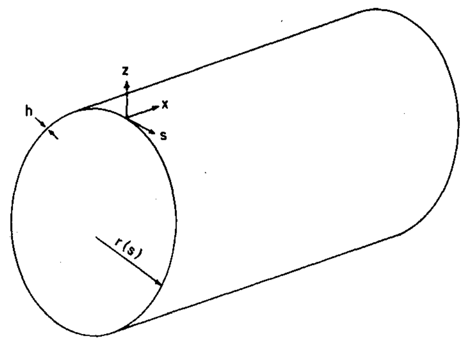

Fig. 1 Shell geometry and coordinate system.

the doubly infinite set of resulting algebraic equations for the Fourier coefficients. For the equations of the Love-Reissner type, Eq. (2), these algebraic equations are given in the Appendix. The resulting equations for the Donnell assumptions are those given in Ref. 1. Suitable convergence was obtained from these by using a digital computer and truncating at $n \leqslant 36$ and $m \leqslant 11$.

Results and Conclusions

By comparing the results for the two cases, Eqs. (2) and (3), it is found that for a given error in the displacements, a much smaller error occurs in the stresses. Although the Donnell solution for axial bending and circumferential membrane stresses are quite accurate, errors in the axial membrane stress as the length of the cylinder increases and errors in the circumferential bending stress as the eccentricity of the oval increases quickly become intolerable.

Hoff[9] has related the accuracy of the Donnell equations for circular cylinders under point and line loadings to the thickness to radius ratio. He lists the error associated with a certain length as a function of this ratio and shows that the error becomes small as the ratio approaches zero. The first of the Donnell assumptions, the neglecting of the shear resultant in the equilibrium of circumferential forces, becomes exact as the thickness to radius ratio approaches zero. Thus, the first approximation governs the breakdown of the Donnell equations for circular cylinders.

By examining the solutions for the oval cylinders under hydrostatic loading, it is found that as the major-to-minor axis ratio of the oval increases, a condition which allows v deflections of increasing magnitude, the error in using the Donnell solution increases. Thus, the second assumption, which ignores the effect of the v displacement in the curvature and twist relations, is seen to break down for oval cylinders. A correlation of this error in predicting stresses vs the major-to-minor axis ratio is given in Fig. 2. Since, for an axially symmetric loaded circular cylinder the Donnell equations are exact, the line $b/a = 1$ in the figure is an asymptote. The 5% and 10% lines indicate the maximum error in the prediction of a particular stress as a percentage of the critical value of σ_x or σ_s. As can be seen, the eccentricity of the cross section as well as the length of the cylinder determine the applicability of the Donnell assumptions for oval cylinders.

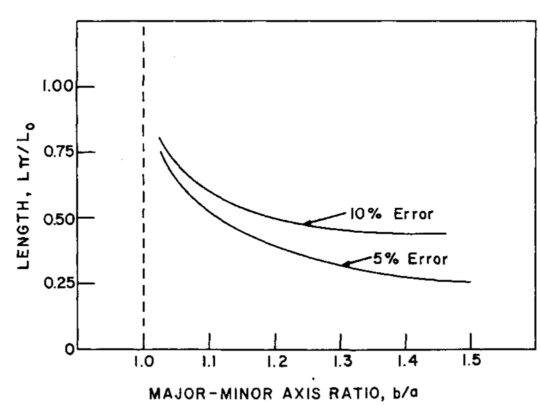

Fig. 2 Error associated with stresses in Donnell solution for hydrostatically loaded oval cylinders.

Although the results are for simply supported boundary conditions, they can be used as a guide for other edge supports. Therefore, for noncircular cylinders, the use of Donnell equations is restricted to short shells, as in the circular case; however, this allowable length is greatly influenced by the eccentricity of the cross section, this length decreasing as the eccentricity increases.

Appendix

The algebraic equations resulting from the substitution of Eqs. (5) and (6) into Eqs. (2) are: for $m = 1,3,5, \ldots$:

$$12s^2(2;1;1)[t^2m^2 + n^2(1-\nu)/2]A_{mn} - 12(0;1;t)[(1+\nu)/2]s^2mnB_{mn} - 12(0;2;1)\nu\xi s^2tmC_{m,n-4} - 12\nu\xi s^2tmC_{m,n+4} - 24\nu s^2tmC_{mn} = 0 \text{ for } n = (0;4;8,12,16,\ldots)$$

$$B_{mo} = 0$$

$$-12[(1+\nu)/2]s^2tmnA_{mn} + (0;1)\pi^2\xi^2\{(n-4)^2 + [(1-\nu)/2]t^2m^2\}B_{m,n-8} + (0;1)4\pi^2\xi\{n^2 - 4n + 8 + [(1-\nu)/2]t^2m^2\}B_{m,n-4} + \{[12s^2 + \pi^2(4 + 2\xi - \xi^2; 4 + 2\xi)][n^2 + [(1-\nu)]/2t^2m^2] + (48;32)\pi^2\xi^2\}B_{mn} + 4\pi^2\xi(1;\xi)\{n^2 + 4n + 8 + [(1-\nu)/2\}t^2m^2]B_{m,n+4} + \pi^2\xi^2\{(n+4)^2 + [(1-\nu)/2]t^2m^2\}B_{m,n+8} + \{\pi^2\xi[(n-4)^3 + t^2m^2(n-4)] + 12s^2\xi n(2;1)\}C_{m,n-4} + \{2\pi^2[n^3 + t^2m^2n] + 24n(s^2;t^2)C_{mn} + \{\pi^2\xi[(n+4)^3 + (1;4)t^2m^2(n+4)] + 12s^2\xi n\}C_{m,n+4} = 0 \text{ for } n = (4;8,12,16,\ldots)$$

$$-12\nu\xi(0;2;1;1)s^2tmA_{m,n-4} - 24\nu s^2tmA_{mn} - 12\nu\xi s^2tmA_{m,n+4} + \xi(0;0;1;1)\{12s^2(n-4) + \pi^2[t^2m^2n + n^3]\}B_{m,n-4} + 2\{12s^2n + \pi^2[n^3 + t^2m^2n]\}B_{mn} + \xi\{12s^2(n+4) + \pi^2[t^2m^2n + n^3]\}B_{m,n+4} + 12\xi^2s^2(0;0;2;1)C_{m,n-8} - 48(0;2;1;1)\xi s^2C_{m,n-4} + \{\pi^2(t^2m^2 + n^2)^2 + 48s^2(y_1;y_2;y_1;y_1)\}C_{mn} + 48\xi s^2C_{m,n+4} + 12\xi^2s^2C_{m,n+8} = (192/m\pi)s^2(1-\nu^2)(1;0;0;0) \text{ for } n = (0;4;8;12,16,20,\ldots)$$

where

$$s = L_o/h, \quad t = L_o/L, \quad y_1 = 1 + (\xi^2/2), \quad y_2 = 1 + \tfrac{3}{4}\xi^2$$

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Symmetrical Bending of Multicore Circular Sandwich Plates

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Introduction

SYMMETRICAL bending of single core circular plates was treated by Reissner,[1] Zaid,[2] and Erickson.[3] Unsymmetrical bending was treated by Kao.[4,5] The bending of multicore circular plates with membrane facings was treated by Stickney and Abdulhadi,[6] where the distinct cores were assumed to possess a common shear angle, in a manner similar to that of Liaw and Little[7] for rectangular plates. Kao and Ross[8] have shown, for multicore beams, that the customarily used common shear angle restriction introduces errors when the cores are not identical.

Analysis of multicore plates suitable for thin or thick cores and without the common shear angle restriction, to our knowledge, does not exist. This Note treats the symmetrical bending for such plates. The governing equations for five-layer plates are derived and solved for arbitrary symmetrical loading and boundary conditions.

Analysis

The facings are distinct, homogeneous and isotropic. The bending and extensional stiffnesses of each facing is taken into account. The facings carry plane stresses while the isotropic cores carry transverse shear stress only. The thickness does not change and the layers do not slip when the plate is loaded.

Let the odd subscripts $i = 1,3,5$ refer to bending layers (facings) and even subscripts $j = 2,4$ refer to shear layers (cores) as shown (Fig. 1). The forces and moments per unit length acting on ith layer are defined as

$$[N_{ir},N_{i\theta}] = \int[\sigma_{ir},\sigma_{i\theta}]dz_i \qquad [M_{ir},M_{i\theta}] = \int[\sigma_{ir},\sigma_{i\theta}]z_i dz_i \quad (1)$$

where $N_{ir}(M_{ir}),N_{i\theta}(M_{i\theta})$ are radial and tangential forces (moments), respectively, and $\sigma_{ir}(\sigma_{i\theta})$ is the radial (tangential) stress. The distance z_i is measured transversely from the center of ith facing.

Hooke's law is used in Eq. (1) leading to

$$\begin{bmatrix} N_{ir} \\ N_{i\theta} \\ M_{ir} \\ M_{i\theta} \end{bmatrix} = \begin{bmatrix} 1 & \nu & 0 & 0 \\ \nu & 1 & 0 & 0 \\ 0 & 0 & -1 & -\nu \\ 0 & 0 & -\nu & -1 \end{bmatrix} \begin{bmatrix} k_i u_{i,r} \\ k_i u_i/r \\ D_i w_{,rr} \\ D_i w_{,r}/r \end{bmatrix} \quad (2)$$

$$k_i = E_iT_i/(1-\nu^2) \qquad D_i = E_iT_i^3/[12(1-\nu^2)] \quad (2a)$$

where ν is common Poisson's ratio, $u_i(w)$ is radial (transverse) displacement, E_i is elastic modulus, and T_i is thickness. Summing moments about the middle of layer five, leads to resultant moment equations

$$M_r = -D(w_{,rr} + \nu w_{,r}/r) - \sum_1^3 \delta_{i5}k_i(u_{i,r} + \nu u_i/r) \quad (3)$$

$$M_\theta = -D(\nu w_{,rr} + w_{,r}/r) - \sum_1^3 \delta_{i5}k_i(\nu u_{i,r} + u_i/r) \quad (4)$$

where $D = D_1 + D_3 + D_5$, and δ_{i5} is the distance between centers of layers five and i. Since bending only is considered, the resultant forces vanish; that is

$$N_{1r} + N_{3r} + N_{5r} = 0 \qquad N_{1\theta} + N_{3\theta} + N_{5\theta} = 0 \quad (5)$$

The equilibrium equations of a differential plate element and Eqs. (3) and (4) are used to obtain an expression for the shear force Q_r

$$Q_r = -D(\nabla^2 w)_{,r} - \sum_1^3 \delta_{i5}k_iS(u_i)$$
$$\nabla^2(\ ) = (\ )_{,rr} + (\ )_{,r}/r \qquad S(\ ) = \nabla^2(\ ) - (\ )/r^2 \quad (6)$$

The core shear stresses τ_{2rz} and τ_{4rz} are obtained from the equilibrium equations of each facing and Eq. (2)

$$\tau_{2rz} = -k_iS(u_i) \qquad \tau_{4rz} = -\sum_1^3 k_iS(u_i) \quad (7)$$

The no-slip conditions between the various layers are

$$u_3 - u_1 = -\delta_{13}w_{,r} + C_2\tau_{2rz} \quad (8)$$

$$u_5 - u_3 = -\delta_{35}w_{,r} + C_4\tau_{4rz} \quad (9)$$

where $C_j = T_j/G_j$ and G_j is shear modulus of jth core. The displacement u_5 is related to u_1 and u_3 as

$$k_5u_5 = -(k_1u_1 + k_3u_3) \quad (10)$$

A set of three governing equations are obtained by use of the equilibrium equations of a plate element and Eqs. (6–10).

$$(1/r)(d/dr)rS(Dw_{,r} + \delta_{15}k_1u_1 + \delta_{35}k_3u_3) = p(r) \quad (11)$$

$$\delta_{13}w_{,r} + C_2k_1S(u_1) - u_1 + u_3 = 0 \quad (12)$$

$$\delta_{35}k_5w_{,r} + (C_4k_1k_5S - k_1)u_1 + (C_4T_3k_5S - k_3 - k_5)u_3 = 0 \quad (13)$$

where $p(r)$ is an arbitrary symmetric load per unit area.

Equations (11–13) are uncoupled to eliminate u_3 and reduce the number of equations to two. The uncoupling process leads to

$$e\frac{dw}{dr} = -fu_1 + b\frac{d}{dr}\frac{1}{r}\frac{d}{dr}(ru_1) + F(p,r) + \frac{A_1r}{2}\left(\log r - \frac{1}{2}\right) + \frac{A_2r}{2} + \frac{A_3}{r} \quad (14)$$

$$\left(\nabla^2 - \frac{1}{r^2} - \lambda^2\right)\left(\nabla^2 - \frac{1}{r^2} - \beta^2\right)u_1 = \left[\alpha F(p,r) - \frac{\eta}{r}\int rp dr\right] + (\alpha A_3 - \eta A_1)\frac{1}{r} + \frac{\alpha A_1 r}{2}\left(\log r - \frac{1}{2}\right) + \frac{\alpha A_2 r}{2} \quad (15)$$

where

$$e = D - \delta_{13}\delta_{35}k_3 \qquad f = \delta_{15}k_1 + \delta_{35}k_3 \quad (15a)$$

$$\alpha = (\delta_{15}k_1 + \delta_{13}k_3)/DC_2C_4k_1k_3k_5 \qquad \eta = \delta_{13}/C_2k_1D \quad (15b)$$

$$F(p,r) = \frac{1}{r}\int r dr \int \frac{dr}{r}\int rp dr \qquad b = \delta_{35}C_2k_1k_3 \quad (15c)$$

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